Maximal green sequences of minimal mutation-infinite quivers

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Spoilers

Theorem. *All minimal mutation-infinite quivers have a maximal green sequence.*

Theorem. *Any cluster algebra generated by a minimal mutation-infinite quiver is equal to its upper algebra.*

Theorem. *The different move-classes of minimal mutation-infinite quivers belong to different mutation-classes (mostly...).*

Quivers and mutations

(Cluster) quiver — directed graph with no loops or 2-cycles.

Mutation μ_k at vertex k:

- Add arrow $i \to j$ for each path $i \to k \to j$
- Reverse all arrows adjacent to k
- Remove maximal collection of 2-cycles

Induced subquiver — obtained by removing vertices.

Quivers and mutations

Quiver Q is **mutation-equivalent** to P if there are mutations taking Q to P.

$\text{Mut}(Q)$ is the **mutation class** of Q containing all quiver mutation-equivalent to Q.

Q is **mutation-finite** if its mutation class is finite. Otherwise it is **mutation-infinite**.

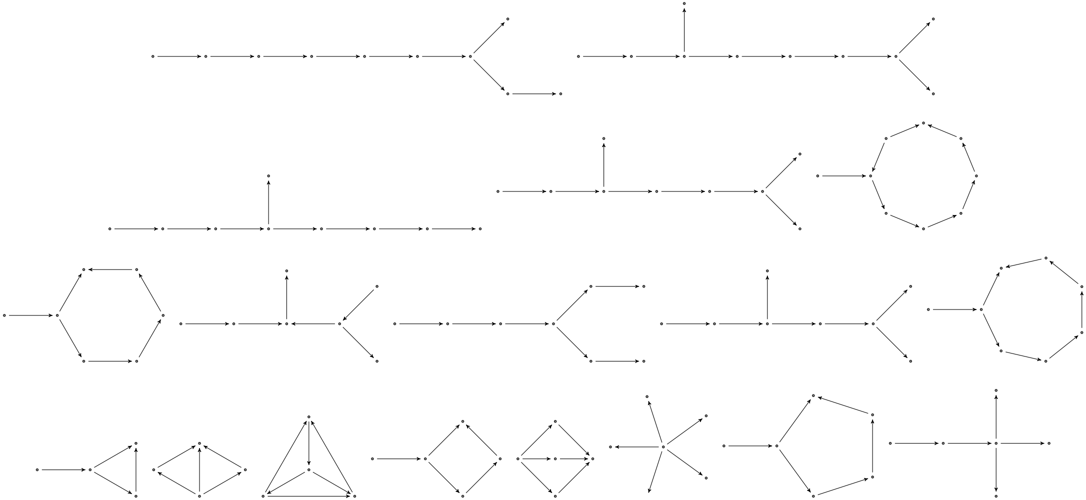
Q is **minimal mutation-infinite** if every induced subquiver is mutation-finite.

MMI classes

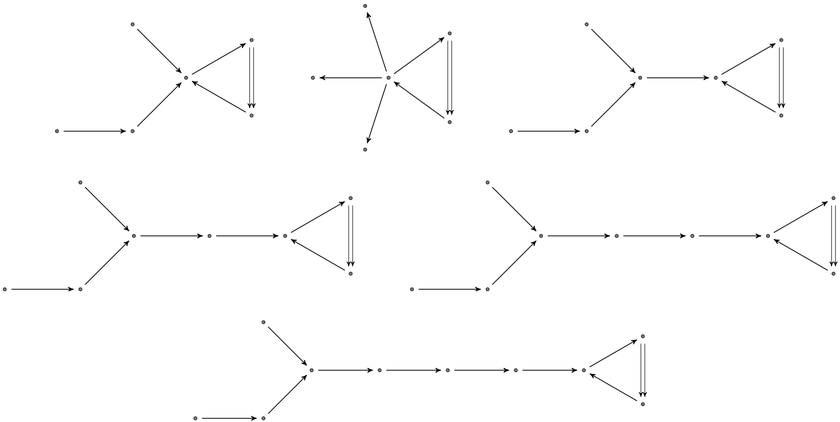
Minimal mutation-infinite quivers classified into move-classes [L '16], with representatives:

- Hyperbolic Coxeter simplex representatives
- Double arrow representatives
- Exceptional representatives

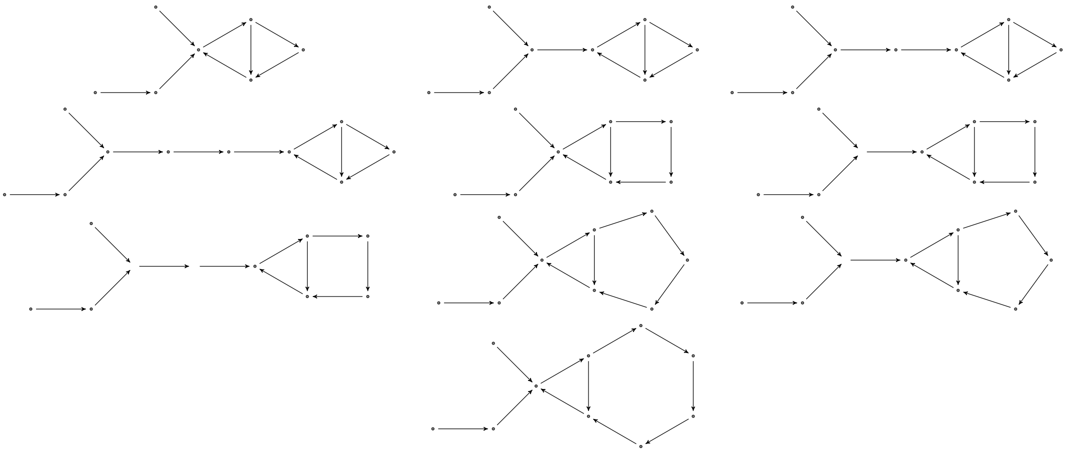
Hyperbolic Coxeter simplex diagrams



Double arrow representatives



Exceptional type representatives



Framed quivers

A **framed quiver** $\widehat{Q}$ is constructed from quiver Q, by adding an additional frozen vertex $\widehat{i}$ for each vertex i in Q and a single arrow $i \to \widehat{i}$.

Red and green

A mutable vertex i in $\hat{Q}$ is **green** if there are no arrows $\hat{j} \to i$.

A mutable vertex i in $\hat{Q}$ is **red** if there are no arrows $i \to \hat{j}$.

Theorem (Derksen-Weyman-Zelevinsky '10). *Any mutable vertex in a quiver is red or green.*

Maximal green sequences

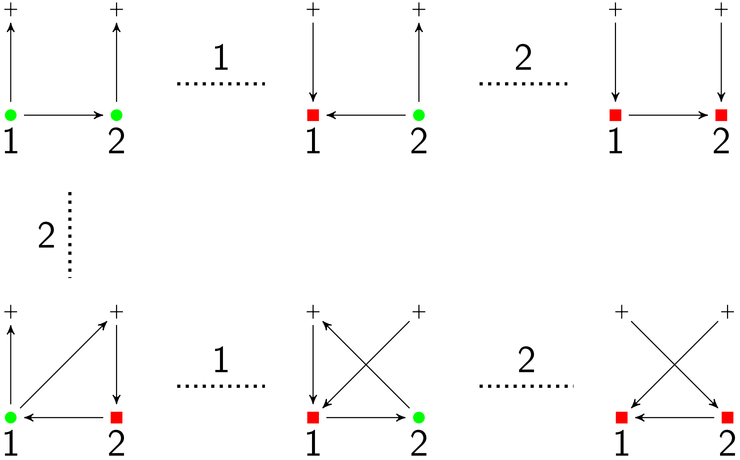
Assume a quiver Q has vertices labelled $(1, \ldots, n)$.

A **mutation sequence** is a sequence of vertices $\boldsymbol{i} = (i_1, \ldots, i_k)$ corresponding to mutating first in vertex i_1, then i_2 and so on.

A **green sequence** is a mutation sequence where every mutation is at a green vertex.

A **maximal green sequence** is a green sequence where every mutable vertex in the resulting quiver is red.

MGS example



Some results

Proposition (Brüstle-Dupont-Perotin '14). *If* $\mathbf{i}$ *is a maximal green sequence for* Q *then* $\mu_{\mathbf{i}}(Q)$ *is isomorphic to* Q*.*

The **induced permutation** of a maximal green sequence is the permutation σ such that $\sigma\big(\mu_{\mathbf{i}}(Q)\big) = Q$.

Theorem (BPS '14). *Any acyclic quiver has a maximal green sequence.*

Proposition (BPS '14). *A quiver* Q *has a maximal green sequence if and only if* Q^{op} *has a maximal green sequence.*

More results

Proposition (Muller '15). *If Q has a maximal green sequence, every induced subquiver has a maximal green sequence.*

Proposition (Muller '15). *Having a maximal green sequence is not mutation-invariant.*

Proposition (Mills '16). *If Q is a mutation-finite quiver, then provided Q does not arise from a once-punctured closed surface and is not mutation-equivalent to the type X_7 quiver, then Q has a maximal green sequence.*

Rotation lemma

Lemma (Brüstle-Hermes-Igusa-Todorov '15). *If* $\boldsymbol{i} = (i_1, i_2, \ldots, i_\ell)$ *is a maximal green sequence of* Q *with induced permutation* σ*, then* $\big(i_2, \ldots, i_\ell, \sigma^{-1}(i_1)\big)$ *is a maximal green sequence for the quiver* $\mu_{i_1}(Q)$ *with the same induced permutation.*

Lemma. *If* $\boldsymbol{i} = (i_1, \ldots, i_{\ell-1}, i_\ell)$ *is a maximal green sequence of* Q *with induced permutation* σ*, then* $\big(\sigma(i_\ell), i_1, \ldots, i_{\ell-1}\big)$ *is a maximal green sequence for the quiver* $\mu_{\sigma(i_\ell)}(Q)$ *with the same induced permutation.*

Direct sums of quivers
[Garver-Musiker '14]

Given two quivers P and Q with k-tuples $(a_1, \ldots, a_k)$ of vertices of P, $(b_1, \ldots, b_k)$ of vertices of Q, the **direct sum**

$$P \oplus_{(a_1,\ldots,a_k)}^{(b_1,\ldots,b_k)} Q$$

is the quiver obtained from the disjoint union of P and Q, with additional arrows $a_i \to b_i$ for each i.

This is a t-**coloured direct sum** if t is the number of distinct vertices in (a_i) and there are no repeated arrows $a_i \to b_j$ added.

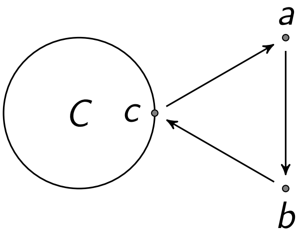
MGS for direct sums

Theorem (GM '14). *If $P = Q \oplus_{(a_1,\ldots,a_k)}^{(b_1,\ldots,b_k)} R$ is a t-colored direct sum, $(i_1, \ldots, i_r)$ is a maximal green sequence for Q, and $(j_1, \ldots, j_s)$ is a maximal green sequence for R, then*

$$(i_1, \ldots, i_r, j_1, \ldots, j_s)$$

is a maximal green sequence for P.

Quivers ending in a 3-cycle

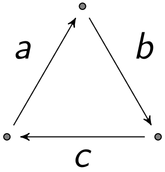


Theorem. *If Q ends in a 3-cycle and C has a maximal green sequence $\boldsymbol{i}_C$, then Q has a maximal green sequence $(b, \boldsymbol{i}_C, a, b)$.*

Rank 3 MMI quivers

and maximal green sequences

$Q_{a,b,c} =$



Proposition (Muller '15). *If* a, b *and* $c \geq 2$ *then* $Q_{a,b,c}$ *does not have a maximal green sequence.*

Proposition. *If any of* a, b *or* c *are* 1*, then* $Q_{a,b,c}$ *has a maximal green sequence.*

Higher ranks

Recall: all mutation-finite quivers have a maximal green sequence, unless they come from a triangulation of a once-punctured closed surface or are mutation-equivalent to X_7.

Lemma. *No minimal mutation-infinite quiver contains a subquiver which does not have a maximal green sequence.*

Corollary. *Every subquiver of a minimal mutation-infinite quiver has a maximal green sequence.*

MMI quivers have MGS

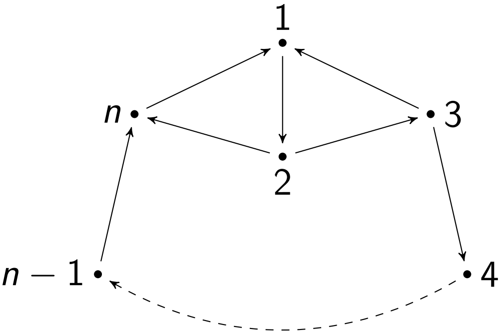
Theorem. *If Q is a minimal mutation-infinite quiver of rank at least 4 then Q has a maximal green sequence.*

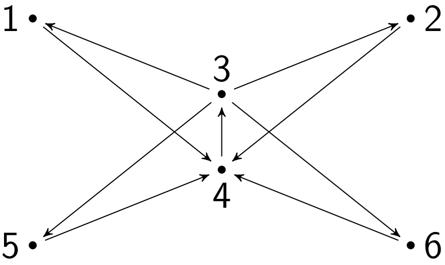
Most have a sink or a source — leaving 192.

Many others are direct sums — leaving 42.

35 of these end in a 3-cycle — leaving 7.

The remaining 7 quivers





Mutation-classes
of MMI move-classes quivers

Moves are sequences of mutations.

Quivers in the same class must be mutation-equivalent.

But does each move-class belong to a different mutation-class?

Tools

Ranks, determinants and acyclics

Rank of the adjacency matrix is mutation-invariant [Berenstein-Fomin-Zelevinsky '05].

Determinant of the adjacency matrix is mutation-invariant.

Whether a quiver is mutation-acyclic — and how many acyclic quivers are in the mutation class [Caldero-Keller '06].

Class	rank(B_Q)	No. Acyclic
4_1	4	6
4_2	2	4
4_3	4	2
4_4	4	1
4_5	4	0
4_6	4	6
5_1	4	8
5_2	4	10
5_3	4	5
5_4	2	5
6_1	4	16
6_2	2	6
6_3	6	10
6_4	6	20
7_1	6	48
7_2	6	12

Class	rank(B_Q)	No. Acyclic
7_3	6	30
7_4	6	28
8_1	8	80
8_2	6	96
8_3	8	14
8_4	8	42
8_5	8	70
9_1	8	219
9_2	8	151
9_3	8	16
9_4	8	55
9_5	8	95
9_6	8	76
10_1	10	225
10_2	8	138

Non mutation-acyclic quivers

How can you prove that a quiver is not mutation-equivalent to an acyclic quiver?

Use the idea of admissible quasi-Cartan companions.

Admissible quasi-Cartans

A **quasi-Cartan companion** of a quiver Q is a symmetric matrix $A = (a_{i,j})$ such that $a_{i,i} = 2$ and $a_{i,j} = |b_{i,j}|$ where $B = (b_{i,j})$ is the adjacency matrix of Q.

A quasi-Cartan companion of Q is **admissible** if for any oriented (resp., non-oriented) cycle Z in Q, there are an odd (resp., even) number of edges $\{i, j\}$ in Z such that $a_{i,j} > 0$.

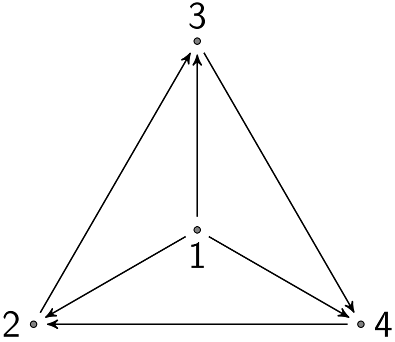
Theorem (Seven '15). *If Q is mutation-acyclic, then Q has an admissible quasi-Cartan companion.*

Admissible quasi-Cartans

How can you prove a quiver does not have an admissible quasi-Cartan companion?

Proposition (Seven '11). *Two admissible companions of a quiver Q can be obtained from one another by a number of simultaneous sign changes in rows and columns.*

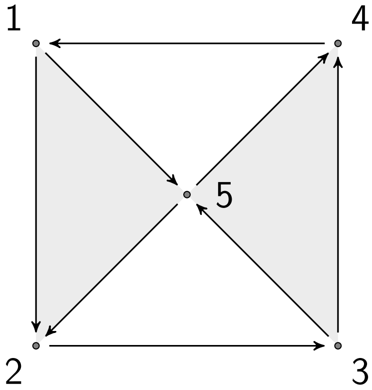
MMI quiver with no admissible companion



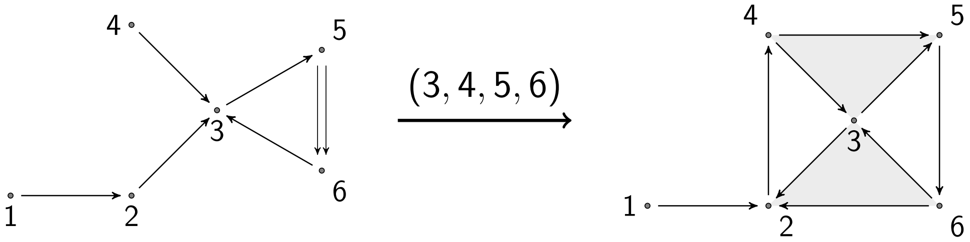
Corollary. *This quiver is not mutation-acyclic.*

Proposition. *Each double arrow move-class contains no acylic quivers.*

Each representative is mutation-equivalent to something which contains:



Example



Same for exceptional classes

Proposition. *Each exceptional move-class contains no acylic quivers.*

But don't know if they belong to different mutation-classes to each other or to the double arrow classes.