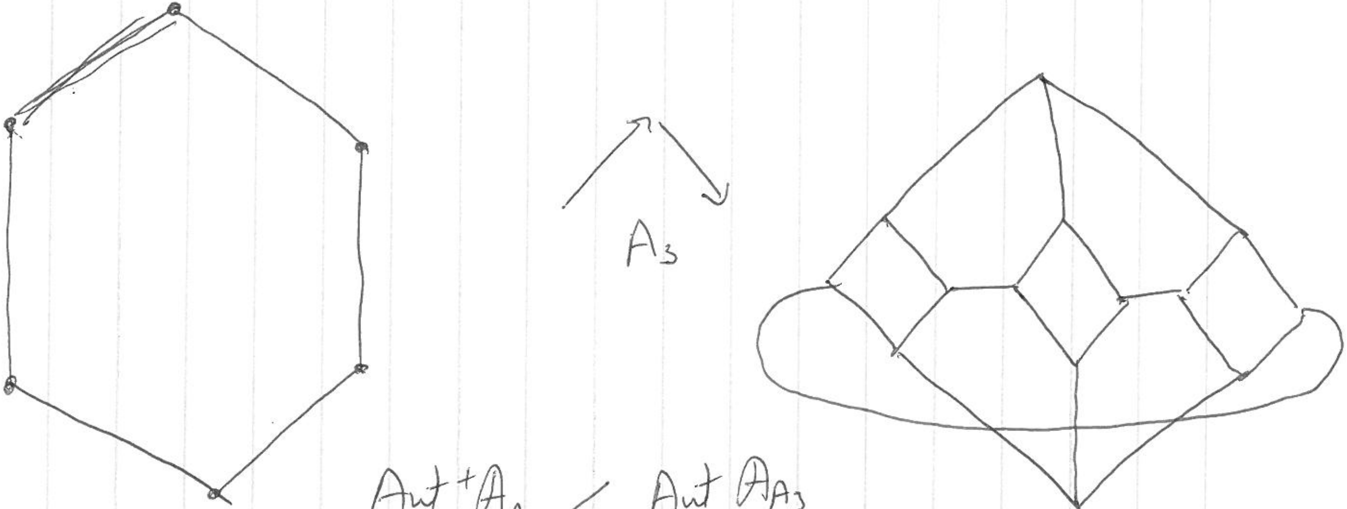
Mapping classes, clusters & combinatorics

<u>Overview</u>

$$\begin{tikzcd}
\text{Surfaces} \arrow[r] \arrow[d, dash] & \text{Tagged triangulations} \arrow[r] & \text{Quiver} \arrow[r] & \text{Cluster algebra} \arrow[r] \arrow[d, dash] & \text{Exchange graph} \arrow[d, dash] \\
\text{MCG} \bowtie & & & \text{Cluster automorphisms} & \text{Graph automorphisms}
\end{tikzcd}$$

Example



$$\mathrm{MCG}_{[\text{illegible}]} = \mathbb{Z}_6 = \mathrm{Aut}^+ \mathcal{A}_{A_3} < \mathrm{Aut}\, \mathcal{A}_{A_3} = \mathrm{Aut}\, \Sigma = \mathbb{Z}_6 \rtimes \mathbb{Z}_2$$

Surface (S,M):

S – orientable surface with (pos. empty) boundary

M – set of marked pts (interior pts = punctures)

$$\mathrm{MCG}(S,M) = \mathrm{Homeos}^{+}(S,M) \Big/ \mathrm{Homeos}^{0}(S,M)$$

Orient-pres homeos fixing M as set

homeos isotopic to id

Fomin-Shapiro-Thurston: add taggings

$$\mathrm{MCG}_{\bowtie}(S,M) = \mathrm{MCG}(S,M) \cancel{\bowtie} \mathbb{Z}_2^{p}$$

where p is no. of punctures.

Cluster algebras

Quiver - oriented graph · no loops
· no 2-cycles

Cluster - $x = \{\beta_1 \dots \beta_n\}$, $\beta_i \in \mathbb{Q}(x_1, \dots x_n) = \mathcal{F}$
β_is algebraically independent

Seed: pair (x, Q)

Mutation at vertex k: μ_k

$$Q \rightsquigarrow Q'$$

$$\beta_i \rightsquigarrow \beta_i \quad i \neq k$$

$$\beta_k \rightsquigarrow \frac{\prod \beta_i + \prod \beta_i}{\beta_k}$$

Mutation class of (x, Q) is set of all seeds obtainable by mutations of (x, Q)

Cluster algebra: subalgebra of $\mathcal{F}$ generated by all rational fns appearing in mutation class

Assem-Schiffler-Schramchenko : 2012
cluster automorphisms

An $\mathcal{F}$-automorphism f is a cluster automorphism if $\exists$ seed (x, Q) in mutation class st

- $f(x)$ appears in seed in mut. class
- Quiver associated to $f(x)$ is either Q or Q^{op}

Auts fixing Q are <u>direct</u> cl. auts
$$\mathrm{Aut}^+ \mathcal{A} < \mathrm{Aut}\, \mathcal{A}$$

<u>Thm (ASS)</u>
$$\mathrm{Aut}\, \mathcal{A} = \begin{cases} \mathrm{Aut}^+ \mathcal{A} \rtimes \mathbb{Z}_2 & \text{if } Q \sim Q^{op} \\ \mathrm{Aut}^+ \mathcal{A} & \text{otherwise} \end{cases}$$

Exchange graph: $\mathcal{E}_{\mathcal{A}}$

vertices - seeds in mut. class

edge $u - v \Leftrightarrow \exists$ mutation μ

$\mu(u) = v$

Graph auts: Permutations σ on vertex set such that

$\exists$ edge $u - v \Leftrightarrow \exists$ edge $\sigma(u) - \sigma(v)$

Thm (Brüstle - Qiu '15) (with some exceptions)

$$MCG_{\bowtie}(S,M) \cong \mathrm{Aut}^{+} \mathcal{A}_{(S,M)}$$

Thm (Chang - Zhu '15) For mutation-finite skew-symmetric (and B_n, C_n $n > 2$)

$$\mathrm{Aut}\,\mathcal{A} = \mathrm{Aut}\,\mathcal{E}_{\mathcal{A}}$$

Skew-symmetrizable case

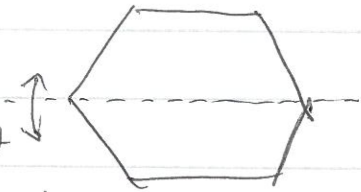
Surfaces $\longrightarrow$ orbifolds

quivers $\longrightarrow$ diagrams

Now $\mathrm{Aut}\,\mathcal{A} \neq \mathrm{Aut}\,\Sigma_{\mathcal{A}}$

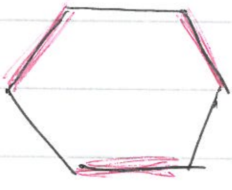
eg

$\xrightarrow{2}$ B_2



reflection does not correspond to cl-aut.

Idea: Add marking to exchange graph



$$\begin{pmatrix} 0 & 1 \\ -2 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} = \text{skew symmetric}$$

Red edge $\longleftrightarrow$ mutation at vertex with symmetrizer 2.

Thm (L) For mutation-finite skew-symmetrizable

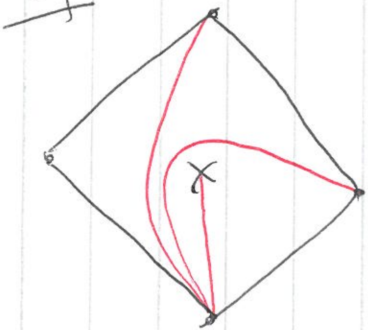
$$\underset{\text{cluster auts}}{\mathrm{Aut}\,\mathcal{A}} \cong \underset{\text{Marked exchange graph auts}}{\mathrm{Aut}\,\hat{\Sigma}_{\mathcal{A}}}$$

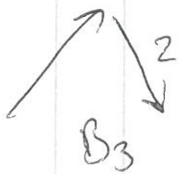
Q: What about orbifold mapping classes?

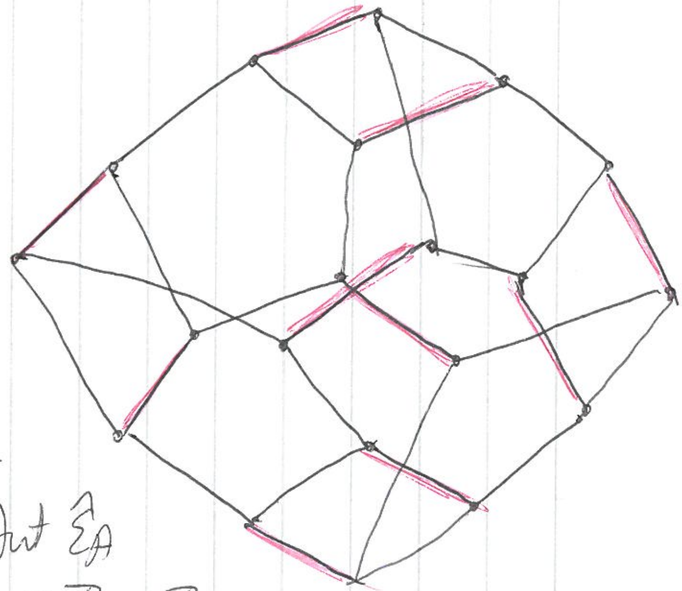
$$\mathrm{Aut}^+\mathcal{A}(\mathcal{O},M) \overset{?}{\cong} \mathrm{MCG}_{\bowtie}(\mathcal{O},M)$$

NB: Markings alternate around squares and hexagons.

Eg







$$\mathrm{MCG}_{\bowtie} = \mathbb{Z}_4 = \mathrm{Aut}^+\mathcal{A} < \mathrm{Aut}\,\mathcal{A} = \mathrm{Aut}\,\hat{\Sigma}_{\mathcal{A}} = \mathbb{Z}_4 \rtimes \mathbb{Z}_2$$