Mapping Classes, Clusters and Combinatories.

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Overview

(Fomin Zelevinsky 02)

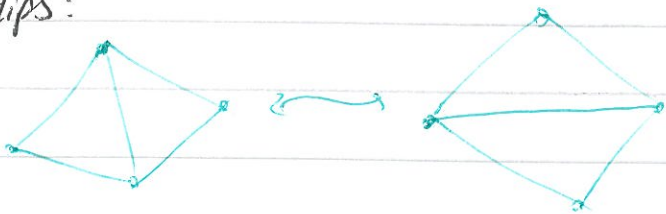
Surfaces → Tri. Cluster algebras

Det Surface with marked points (S, M) is an orientable surface S with (possibly empty) boundary and marked points M such that each ∂-component contains at least 1 marked pt. Interior marked pts = punctures.

Det Triangulation by covering S with triangles with vertices in M

Self folded triangles.

Fact: Any two triangulations of (S, M) differ by a number of triangle flips:



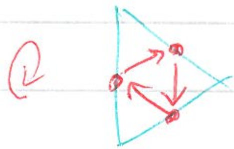
Introduce taggings (Fomin-Shapiro-Thurston, 08)

Then all edges can be flipped.

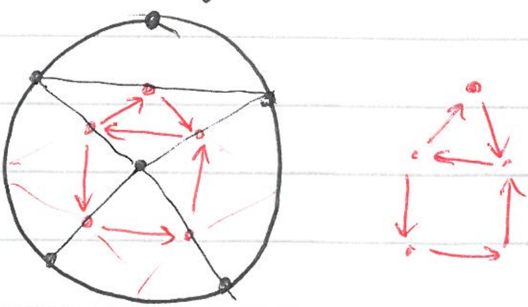
Quivers:

Def A quiver is an oriented graph
with
- no loops
- no 2-cycles

Given triangulation construct quiver:



eg. Punctured pentagon:



Overview

Surface $\rightarrow$ Tri. $\rightarrow$ Quiver $\rightarrow$ Cl. alg

Cluster algebras

Fix ambient field $\mathcal{F} = \mathbb{C}(x_1 \ldots x_n)$
field of rational fns on n vars.

n = no. of vars = no. of vertices in quiver

Def: Seed is a pair (x, Q)
Q – quiver
$x = \{\beta_1 \ldots \beta_n\}$ where $\beta_i \in \mathcal{F}$
~~Ku~~ are algebraically independent.

Def Mutation at kth vertex:

$$\mu_k(x, Q) = (x', Q')$$

where Q' is given by combinatorial rules and $\beta_i' = \beta_i$ for $i \neq k$

$$\beta_k' = \frac{\prod \beta_i + \prod \beta_j}{\beta_k}$$

Def Cluster alg is subalg of $\mathcal{F}$ gen. by all possible β_i.

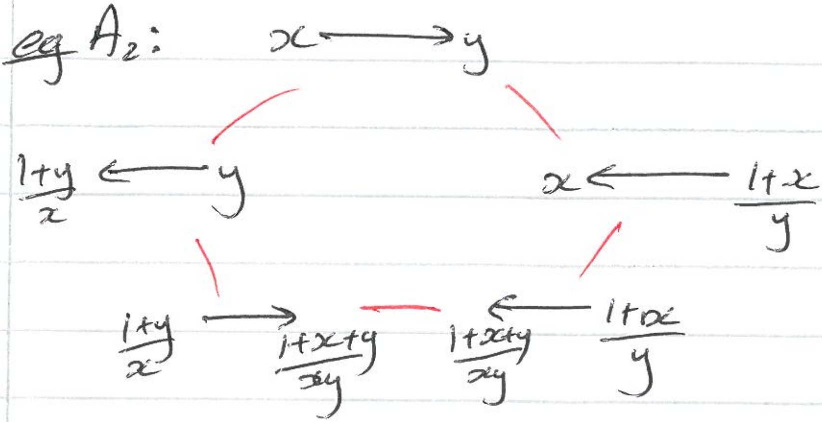
Fact: Mutation is involution:

$$(x, Q) \underset{\mu_k}{\overset{\mu_k}{\rightleftarrows}} (x', Q')$$

Construct exchange graph:

$$\left.\begin{array}{l} \text{vertices — seeds} \\ \text{edges — mutations} \end{array}\right\} \text{up to permutation.}$$

eg A_2:



Overview

$$\begin{tikzcd}
S \arrow[r] \arrow[d] & T \arrow[r] \arrow[dl, dashed] & Q \arrow[r] & \text{cl. alg} \arrow[r] & \text{Exch gr.} \\
\text{(tagged) MCG} & & & &
\end{tikzcd}$$

Def MCG: group of orientation preserving diffeomorphisms of (S,M) which fix a set of marked pts M. up to diffeos. isotopic to identity:

$$MCG(S,M) = \mathrm{Diffeo}^+(S,M) / \mathrm{Diffeo}^0(S,M)$$

$$\underline{\text{Tagged MCG}}: \quad MCG(S,M) \ltimes \mathbb{Z}_2^{|P|}$$
$$MCG_{\bowtie}(S,M) = MCG(S,M) \ltimes \mathbb{Z}_2^{|P|}$$

where each $\mathbb{Z}_2$ corresponds to a puncture p & acts by changing the tagging at p.

Overview

$$\begin{tikzcd} S \arrow[r] \arrow[d] & T \arrow[r] \arrow[dl, dashed] & Q \arrow[r] & A \arrow[r] & G \arrow[d] \\ MCG_{\bowtie} & & & & \text{Graph automorphisms} \end{tikzcd}$$

Def Graph automorphism of G is a permutation σ of the vertices of G such that

$$\exists \text{ edge } u - v \iff \exists \text{ edge } \sigma(u) - \sigma(v)$$

Overview

$$\begin{tikzcd} S \arrow[r] \arrow[d, "E"] & T \arrow[r] & Q \arrow[r] & A \arrow[r] \arrow[d] & G \arrow[d] \\ MCG_{\bowtie} & & & \text{cluster automorphisms} & \mathrm{Aut}\, G \end{tikzcd}$$

(Assem, Schiffler, Shramchenko, 12)

Def An f-automorphism is a cluster automorphism of A if it fixes the cluster structure:

so $\exists$ seed (x, Q) such that

- $f(x)$ is a cluster in some seed of A
- $f(Q)$ is isomorphic to Q or Q^{op}

Fact: (ASS) Cl. auts form a group: $\mathrm{Aut}\, A$

Those which take $Q \rightsquigarrow Q$

form a subgroup: direct cluster automorphisms $\mathrm{Aut}^+ A$.

Overview

$$
\begin{tikzcd}
S \arrow[r] \arrow[d] & T \arrow[r] \arrow[dl] & Q \arrow[r] & \mathcal{A} \arrow[r] \arrow[d] & G \arrow[d] \\
MCG_{\bowtie} \arrow[drr, phantom, "\simeq"] & & & \operatorname{Aut} \mathcal{A} \arrow[r, phantom, "\simeq"] & \operatorname{Aut} G \\
& & \operatorname{Aut}^+ \mathcal{A} \arrow[ur] & &
\end{tikzcd}
$$

<u>Thm</u> (Brüstle–Qiu, 15)

For (most) surfaces (S, M)

$$MCG_{\bowtie}(S, M) \simeq \operatorname{Aut}^+ \mathcal{A}.$$

<u>Thm</u> (ASS) $\operatorname{Aut} \mathcal{A}$ is either equal to $\operatorname{Aut}^+ \mathcal{A}$ or $\operatorname{Aut}^+ \mathcal{A} \rtimes \mathbb{Z}_2$

↑

When Q and Q^{op} are mutation equivalent.

<u>Thm</u> (Chang–Zhu, 15)

For surface cluster algebras

$$\operatorname{Aut} \mathcal{A} \simeq \operatorname{Aut} G.$$

Overview

$$\begin{tikzcd}
 & \mathrm{MCG}_{\bowtie} \arrow[r] \arrow[dl, dash] \arrow[d, dash] & \mathrm{Aut}^+ A \arrow[r] & \mathrm{Aut}\, A \arrow[r] \arrow[d, dash] & \mathrm{Aut}\, G \arrow[d, dash] \\
S \arrow[r] & T \arrow[r] & Q \arrow[r] & A \arrow[r] & G
\end{tikzcd}$$

$$\begin{tikzcd}
O \arrow[r] \arrow[d, dash] & T \arrow[r] & D \arrow[r] \arrow[d, dash] & A \arrow[r] & G \\
\text{orbifold} & & \text{diagram} & &
\end{tikzcd}$$

For orbifold case: no longer have
$\mathrm{Aut}\, A \cong \mathrm{Aut}\, G$.

↑

too big.

My work: Add a marking to exchange graph, so that auts fixing the marking give cluster auts.

Thm (L): $\mathrm{Aut}\, A \cong \mathrm{Aut}\, \widehat{G}$